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A Complementarity Problem in Mathematical Programming in Banach Space

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An existence and uniqueness theorem for the nonlinear complementarity problem over arbitrary closed convex cones in a reflexive real Banach space is established.

1. INTRODUCTION AND STATEMENT OF THE THEOREM

Let B be a reflexive real Banach space and let B^* be its dual. Let the value of $u \in B^*$ at $x \in B$ be denoted by (u, x). Let C be a closed convex cone in B with the vertex at 0. The polar of C is the cone C^*, defined by

$$C^* = \{u \in B^*: (u, x) \geqslant 0 \text{ for each } x \in C\}.$$

For any $e \in C^*$ and each $r > 0$ we write

$$D_r(e) = \{x \in C: 0 \leqslant (e, x) \leqslant r\},$$
$$D_r^0(e) = \{x \in C: 0 \leqslant (e, x) < r\},$$
$$S_r(e) = \{x \in C: (e, x) = r\}.$$

A mapping $T: C \to B^*$ is said to be monotone if $(Tx - Ty, x - y) \geqslant 0$ for all $x, y \in C$ and strictly monotone if strict inequality holds whenever $x \neq y$. T is said to be α-monotone if there is a strictly increasing function $\alpha: [0, \infty) \to [0, \infty)$ with $\alpha(0) = 0$ and $\alpha(r) \to \infty$ as $r \to \infty$ such that $(Tx - Ty, x - y) \geqslant \|x - y\| \, \alpha(\|x - y\|)$ for all $x, y \in C$. In particular T is strongly monotone if $\alpha(r) = kr$ for some $k > 0$. T is said to be hemicontinuous on C if for all $x, y \in C$, the map $t \to T(ty + (1 - t)x)$ of $[0, 1]$ to B^* is continuous when B is endowed with the weak* topology. T is said to be bounded if T maps bounded subsets of C into bounded subsets of B^*.

The purpose of this note is to prove the following existence and uniqueness theorem for the nonlinear complementarity problem.

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THEOREM. *Let $T: C \to B^*$ be bounded, hemicontinuous and strictly monotone such that there is an $x \in C$ with $Tx \in C^*$. Then there is a unique x_0 such that*

$$x_0 \in C, Tx_0 \in C^* \qquad \text{and} \qquad (Tx_0, x_0) = 0. \tag{1.1}$$

This work has been motivated by the work of Bazaraa, Goode and Nashed [1], who have proved the same result under the assumption that the operator T is bounded, hemicontinuous and α-monotone. Our result is different from those obtained in [1]. We have only assumed strict monotonicity instead of α-monotonicity (which is stronger) but we have made a feasibility assumption. Our result contains the results obtained in [9] as particular cases.

Several authors have discussed the nonlinear complementarity problem in finite dimensional spaces; see, for example [2, 3, 6, 7]. Besides the work of Bazaraa, Goode and Nashed [1], the nonlinear complementarity problem in infinite dimensional spaces appear in [4, 8, 9].

2. PROOF OF THE THEOREM

The following result, which will be needed in the sequel, is a special case of Theorem A of Mosco [5]. See also [1].

LEMMA (Mosco). *Let $T: C \to B^*$ be bounded, hemicontinuous and strictly monotone and let K_r be a family of nonempty closed convex sets in C. Then for each r there is a unique $x_r \in K_r$ such that $(Tx_r, z - x_r) \geqslant 0$ for all $z \in K_r$.*

Proof of the theorem. For any $e \in C^*$ and each $r > 0$, $D_r(e)$ is clearly convex. Furthermore the function $f: C \to \mathbb{R}$ defined by $f(z) = (e, z)$ is obviously continuous. But $D_r(e) = f^{-1}[0, r]$. Hence $D_r(e)$ is closed. Therefore it follows from Lemma (Mosco) that for each $r > 0$, there is a unique $x_r \in D_r(e)$ such that

$$(Tx_r, z - x_r) \geqslant 0 \qquad \text{for all } z \in D_r(e). \tag{2.1}$$

Since $0 \in D_r(e)$ it follows that $(Tx_r, x_r) \leqslant 0$ for all r. If there exist $e \in C^*$ and $r > 0$ such that $x_r \in D_r^0(e)$, then there is some $\lambda > 1$ such that $\lambda x_r \in S_r(e) \subset D_r(e)$. Then we have from (2.1) that $(Tx_r, x_r) \leqslant (Tx_r, \lambda x_r) = \lambda(Tx_r, x_r)$. Since $(Tx_r, x_r) \leqslant 0$, this is impossible unless $(Tx_r, x_r) = 0$; thus x_r is a solution to (1.1). Now assume that $x_r \in S_r(e)$ for all $e \in C^*$ and all

$r > 0$. By the hypothesis there is an $x \in C$ with $Tx \in C^*$. Set $e = Tx$. Choose $r > (Tx, x) \geqslant 0$. Now $x \in D_r^0(Tx)$ and since T is monotone we have

$$\begin{aligned} &(Tz, z - x) \geqslant (Tx, z - x) > 0 \text{ for all } z \in S_r(Tx). \\ &\text{But } x_r \in S_r(Tx) \text{ and hence } (Tx_r, x_r - x) > 0. \end{aligned} \tag{2.2}$$

Since $x \in D_r^0(Tx) \subset D_r(Tx)$, it follows from (2.1) that $(Tx_r, x - x_r) \geqslant 0$, i.e., $(Tx_r, x_r - x) \leqslant 0$. Since this contradicts (2.2), the assumption "that $x_r \in S_r(e)$ for all r" has thus been shown not to hold when $e = Tx$. Thus the proof of the theorem is reduced to the previous case (where there exists $e \in C^*$ and $r > 0$ such that $x \in D_r^0(e)$). Since T is strictly monotone, the solution is unique and this completes the proof. Note that the assertion "$(Tx, z - x) > 0$" is true for $z \in S_r(Tx)$ and is not true for all $z \in C$.[1] This follows from the following example which was very kindly suggested by the referee.

Let $B = \mathbb{R}^2$. Let $C = \{(x, y) \in \mathbb{R}^2 : x \geqslant 0 \text{ and } y = 0\}$, so that $C^* = \{(x, y) \in \mathbb{R}^2 : x \geqslant 0\}$. Let $T: C \to B^*$ be given by $T((x, 0)) = (x, 0)$. If $x, z \geqslant 0$, then $(x, 0) \in C$, $(z, 0) \in C$ and $T((x, 0)) = (x, 0) \in C^*$. However, $(T((x, 0)), (z, 0) - (x, 0)) = x(z - x) < 0$ if $z = 0$ and $x > 0$.

As another example we may consider: $B = \mathbb{R}$. $C = \{x : x \geqslant 0\}$, so that $C = C^*$. Let $T: C \to B^*$ be given by $T(x) = x$. If $x, z \geqslant 0$, then $x \in C$, $z \in C$ and $Tx = x \in C^*$. However, $(Tx, z - x) = x(z - x) < 0$ if $z = 0$ and $x > 0$. Note that the assumption that there is an $x \in C$ with $Tx \in C^*$ can fail to be satisfied. For example, take $B = \mathbb{R}$. Let $C = \{x \in \mathbb{R} : x \geqslant 0\}$, so that $C = C^*$. Define $T: C \to \mathbb{R}$ by $Tx = -1/(1 + x)$. Then although T is strictly monotone, there is no $x_0 \in C$ such that $Tx_0 \in C^*$ and hence there is no $x_0 \in C$ which satisfies (1.1).

If we take $B = \mathbb{R}$, $C = \{x \in \mathbb{R} : x \geqslant 0\}$, and $T: C \to \mathbb{R}$ defined by $Tx = x - 1$. Then $(Tx, x) = 0$ implies that $x = 0$ or $x = 1$. But $x = 1$ is a solution of (1.1); $x = 0$ is not a solution since $T0 = -1 \notin C^*$. Note that T is strictly monotone and the feasibility assumption is satisfied in this case and hence (1.1) has a unique solution $x = 1$.

Finally we conclude the paper with the following corollaries:

COROLLARY 1 [Theorem 2, 9]. *Let $T: C \to B^*$ be bounded, hemicontinuous and strictly monotone such that either $T0 = 0$ or $T0 \in C^*$. Then $x_0 = 0$ is the unique solution to* (1.1).

Proof. Since the hypothesis of the theorem is satisfied by $x = 0$, it now

[1] In the original version of the paper the authors wrote that $(Tx, z - x) \geqslant 0$ for all $z \in C$. This resulted in a gap in the proof of the main theorem which was pointed out by the referee. The authors wish to express their gratitude to the referee for this and also for his suggestions which improved the presentation of the paper.

follows from the proof of the theorem that for each $r > 0$, x_r is a solution to (1.1). Since T is strictly monotone, (1.1) can have at most one solution, say x_0. Therefore $\|x_0\| = \|x_r\| \leqslant r$ for each $r > 0$. So $x_0 = 0$ and this completes the proof.

COROLLARY 2 [Theorem 1, 9]. *Let* $T: C \to B^*$ *be bounded, hemicontinuous and strictly monotone and let there be a constant* $k > 0$ *such that* $\|Tx\| \leqslant k\|x\|$ *for every* $x \in C$. *Then* $x_0 = 0$ *is the unique solution to* (1.1).

Proof. Since $\|Tx\| \leqslant k\|x\|$ for every $x \in C$ and since $0 \in C$, it follows that $T0 = 0$. Now the result follows from Corollary 1.

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